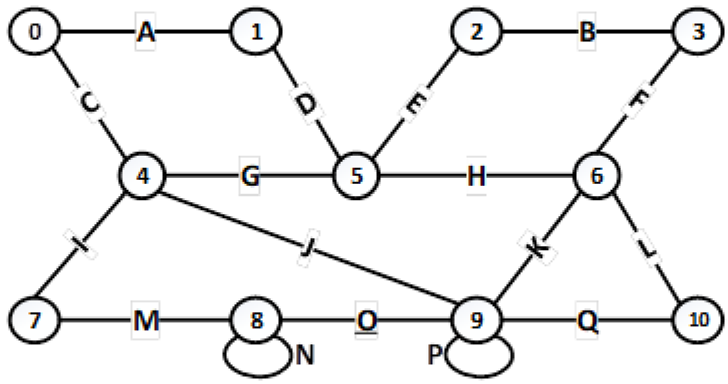
PART A – GRAPH THEORY – 25 MARKS

A1 Euler and Hamiltonian Circuits (6 marks)



a) This graph has many **Euler** circuits starting at vertex 0. Every Euler circuit can be traversed in two directions, starting either with edge A and finishing with C, or starting with edge C and finishing with A. Similarly, any subcircuit of an Euler circuit can be traversed in two directions. Here are some of the most commonly found circuits that start with edge A (there are more):

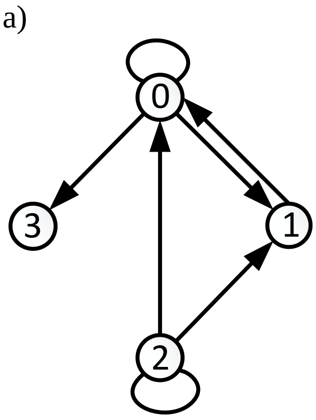
- ADEBFLQPKHGJONMIC
- ADEBFHGJKLQPONMIC
- ADEBFLQPONMIJKHGC
- ADEBFLQKHGJPONMIC

b) This graph has two Hamiltonian circuits starting at vertex 0:

- ADEBFLQOMIC and its reverse CIMOQLFBEDA

A2 Matrices in Graph Theory (5 marks)

a)

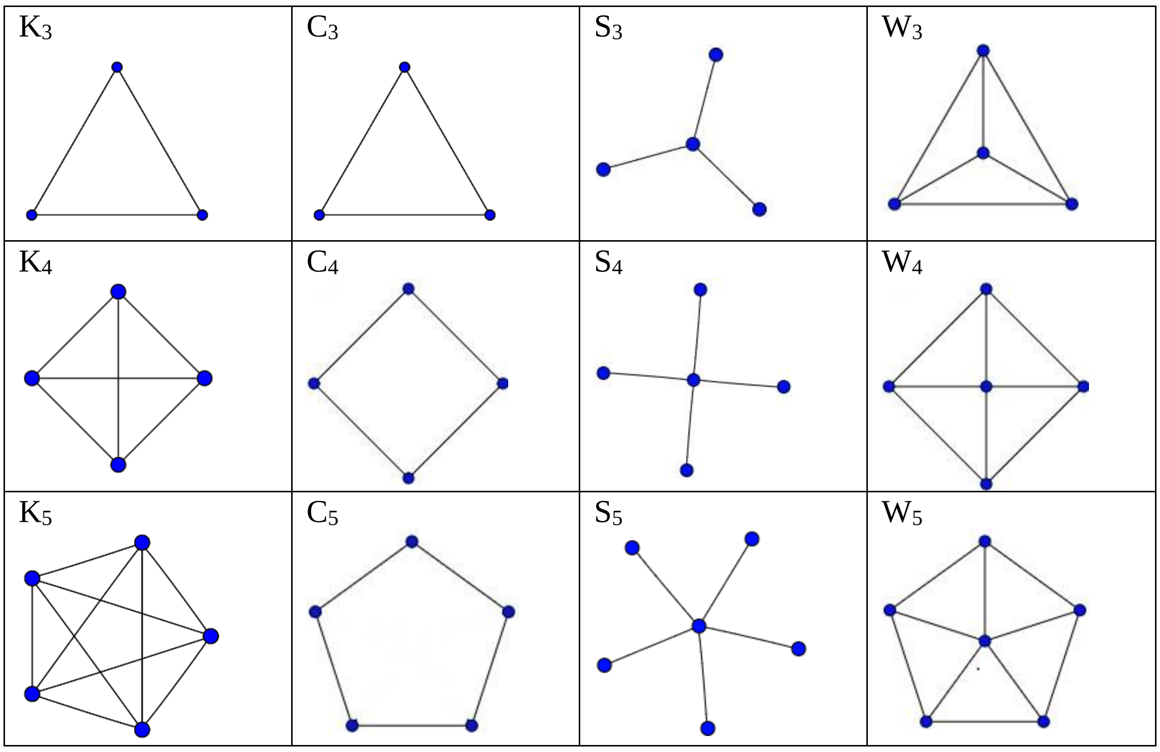


b)

i \ j	0	1	2	3
0	**2**	**1**	**0**	**1**
1	**1**	**1**	**0**	**1**
2	**3**	**2**	**1**	**1**
3	**0**	**0**	**0**	**0**

1. <u>A3 Simple Graphs (14 marks)</u>

a)



b) W_n is the graph formed by combining C_n and S_n:

i.e. $V(W_n)=V(S_n)$ and $E(W_n)=E(S_n)\cup E(C_n)$

i.e $W_n - S_n = C_n$

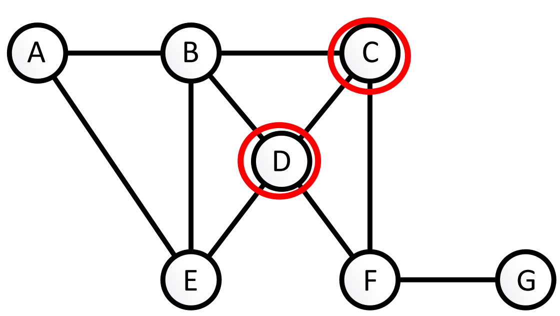
c) For the graph G underneath:

i. Circle all the centers of G on the diagram

ii. Radius $\rho(G) = 2$

iii. Diameter $\delta(G) = 4$

iv. Give one of the walks (i.e. list the vertices of the walk in order) that is the length of this diameter: ABDFG, or ABCFG, or AEDFG



d) For the special graphs defined in a) when $n \geq 3$:

Graph G	K_n	C_n	S_n	W_n
Radius $\rho(G)$	1	$\left\lfloor \frac{n}{2} \right\rfloor$	1	1
Diameter $\delta(G)$	1	$\left\lfloor \frac{n}{2} \right\rfloor$	2	$\begin{cases} 1 \text{ when } n = 3 \\ 2 \text{ when } n > 3 \end{cases}$

PART B – SEQUENCES, RECURRENCE RELATIONS – 10 MARKS

Given the sequence a_n defined with the recurrence relation:

$a_0 = 3$
$a_k = 5a_{k-1} + 2k$ for $k \geq 1$

2. Terms of the Sequence (5 marks)

$a_1 = 5.3 + 2.1$
$a_2 = 5(5.3 + 2.1) + 2.2 = 3.5^2 + 5.2.1 + 2.2$
$a_3 = 5(3.5^2 + 5.2.1 + 2.2) + 2.3 = 3.5^3 + 5^2.2.1 + 5^1.2.2 + 5^0.2.3$
$a_4 = 5(3.5^3 + 5^2.2.1 + 5^1.2.2 + 5^0.2.3) + 2.4 = 3.5^4 + 5^3.2.1 + 5^2.2.2 + 5^1.2.3 + 5^0.2.4$
$a_5 = 5(3.5^4 + 5^3.2.1 + 5^2.2.2 + 5^1.2.3 + 5^0.2.4)+ 2.5 = 3.5^5 + 5^4.2.1 + 5^3.2.2 + 5^2.2.3 + 5^1.2.4 + 5^0.2.5$

3. Iteration (5 marks)

$a_n = 3.5^n + 2\sum_{i=0}^{n-1} 5^i(n-i)$

PART C – INDUCTION – 15 MARKS

Given the sequence b_n defined recursively as:

$b_0 = 3$, $b_1 = 7$
$b_k = 3b_{k-1} - 2b_{k-2}$ for $k \geq 2$

You will now prove by **strong induction** that a solution to this sequence is $b_n = 2^{n+2} - 1$.

1. Problem Statement (2 marks)

The conjecture being proved is expressed symbolically in the form $\forall n \in D, P(n)$, where:

a) (1 mark) $D = \mathbb{N}$
b) (1 mark) P(n) is: $b_n = 2^{n+2} - 1$

2. Base Cases (4 marks)

- When n=0, $2^{n+2} - 1 = 2^2 - 1 = 3 = b_0$
- When n=1, $2^{n+2} - 1 = 2^3 - 1 = 7 = b_1$

3. Inductive step setup (3.5 marks)

- (2 marks) State the assumption in the inductive step and identify the inductive hypothesis.
  Assume that some $k \geq 1$ is such that $\forall m \in \{0,\ldots,k\}$ $b_m = 2^{m+2} - 1$ ← IH: Inductive Hypothesis
- (1.5 marks) State what you will be proving in the inductive step.
  We will prove P(k+1), i.e. $b_{k+1} = 2^{(k+1)+2} - 1 = 2^{k+3} - 1$

4. Remainder of Inductive step (5.5 marks).

Since $k \geq 1$ then $k+1 \geq 2$ and the recurrence relation applies to k+1:
$b_{k+1} = 3b_k - 2b_{k-1}$
$k \leq k$ and $k-1 \leq k$ and therefore the inductive hypothesis applies to them:
$b_k = 2^{k+2} - 1$ and $b_{k-1} = 2^{k+1} - 1$
Therefore:
$b_{k+1} = 3b_k - 2b_{k-1} = 3(2^{k+2} - 1) - 2(2^{k+1} - 1)$ By IH
$= 3.2^{k+2} - 3 - 2.2^{k+1} + 2$
$= 3.2^{k+2} - 1 - 2^{k+2}$
$= 2.2^{k+2} - 1$
$= 2^{k+3} - 1$ Algebra
QED